

Three loop renormalization of 3-quark operators in QCD

J.A. Gracey,
 Theoretical Physics Division,
 Department of Mathematical Sciences,
 University of Liverpool,
 P.O. Box 147,
 Liverpool,
 L69 3BX,
 United Kingdom.

Abstract. We compute the three loop $\overline{\text{MS}}$ anomalous dimension for the 3-quark operator corresponding to the proton. This requires the treatment of γ^5 within dimensional regularization as well as evanescent operators generated through the renormalization. We extend the Larin scheme for γ^5 to a mixing matrix of finite renormalization constants chosen so that chiral symmetry is manifest in four dimensions. We also provide the finite part of the Green's function at two loops where the operator is inserted at zero momentum in a quark 3-point function in an arbitrary linear covariant gauge in order to assist with the lattice measurement of the same quantity. The renormalization of the generalized 3-quark operators in the scheme devised by Krämer and Manashov is extended to three loops and the anomalous dimensions for the $(\frac{1}{2}, 0)$, $(\frac{3}{2}, 0)$ and $(1, \frac{1}{2})$ spin operators with various chiralities are also given.

1 Introduction.

In quantum field theory baryons are represented by 3-quark operators where the operators are chosen so as to have the same discrete and continuous symmetries as the observed states in nature guided by the quark model. For instance, protons are built from two up and one down quark fields with an overall spin of spin $\frac{1}{2}$. With suitable choices of handedness for these three fields one can construct a 3-quark operator with the correct J^{PC} values for the proton. While in nature these quarks are in general massive and confined, for some theoretical studies of such hadronic states one can consider them to be built from chiral or massless fields. This is an appropriate approximation in the high energy limit where the masses are small when compared to the momentum scale. Thus the leading twist operators will dominate any high energy analysis, [1, 2, 3]. At low energies the internal structure of the baryons and in particular protons, can be probed by measuring the structure functions and their moments. For the latter the field theoretic quantities of relevance are 3-quark operators involving covariant derivatives, [1, 2, 3, 4, 5, 6]. Knowledge of the proton structure can assist with understanding the strong interaction in the infrared regime and hence how quarks condense or hadronize to form the nucleon states we see in nature. In quantum field theory the use of the Quantum Chromodynamics (QCD) Lagrangian provides us with a forum to study these different aspects of hadrons. At high energy one can apply perturbation theory and calculate order by order in the strong coupling constant which is assumed to be small in this region. However, while this provides one with the renormalization group evolution of the operators with scale, it is the non-perturbatively measured operator matrix element which gives the information relating to the baryon structure. Such matrix elements cannot be determined perturbatively for the purpose of extracting physical results. Instead they are measured non-perturbatively using lattice gauge theory where the spacetime is discretized. Although this involves the use of large computers to handle the huge numerical calculations, there are various technical issues underlying the process. One of these resides in making accurate measurements through credibly small error bars. Moreover, one needs to have contact with the continuum structure of the same quantities once the lattice regularization is lifted. Indeed it should be the case that when low energy estimates from the lattice are extrapolated to the high energy continuum evaluation there should be reasonable agreement.

To this end over a period of years there has been ongoing comparison of lattice computations with high energy perturbative expressions. The latter are determined in the chiral limit, usually in the $\overline{\text{MS}}$ scheme, to as high a loop order as is computationally possible. Indeed with the current loop calculation technology this invariably means *three* loops through the use of the MINCER package, [7, 8]. As highlights of this bridge we mention quark current renormalization for zero momentum operator insertions, [9, 10], and more recently at non-zero momentum insertion, [11, 12, 13, 14, 15]. The latter are known at two loops at the symmetric subtraction point for the $\overline{\text{MS}}$ scheme. However, in both cases results have been determined in the regularization invariant (RI), [16, 17], class of schemes which includes RI' and RI'/SMOM, [11], where SMOM indicates momentum subtraction at the symmetric point. These are schemes which are devised for lattice regularization in order to minimize the use of derivatives, which is computationally intense, in extracting renormalization constants as well as amplitudes. Both sets of schemes have continuum analogues, [9, 10]. Building on this project it is natural now to turn from quark-antiquark operators representing mesonic physics to 3-quark operators for baryon problems. Therefore, it is the purpose of this article to compute the Green's function for the zero momentum insertion of the 3-quark operator representing a proton in the chiral limit to two loops in the $\overline{\text{MS}}$ scheme. In addition to assist with running to high energy we will compute the operator anomalous dimension to *three* loops in the same scheme. Both results will therefore be important for matching lattice results of the same Green's function in the high energy limit. For the operators which we

consider here, we restrict ourselves to the $\overline{\text{MS}}$ scheme and will not introduce any variant of the regularization invariant class of renormalization schemes. Though from the results compiled here it is possible to define such schemes for the 3-quark operator renormalization and then convert to the $\overline{\text{MS}}$ scheme. However, one reason for not choosing to explicitly include, say, RI' results rests in the fact that as noted in [14] the definition of such schemes is not unique for Lorentz structure beyond the simplest quark mass operator. Indeed in the tensor current case a scheme alternative, [14], to the original one of [11, 13] appeared to converge quicker at two loops. Only a three loop calculation would give more insight into this.

In more detail we will evaluate the two loop Green's function for an arbitrary linear covariant gauge at a point where the squared momenta of the three external quark legs are all equal to the same non-zero value. Hence, there is no zero momentum external quark leg. This is appropriate since a zero momentum quark would be difficult to incorporate on the lattice. Several lattice studies of 3-quark operators and their low moments for similar Green's functions deserve mention. In [18] an initial one loop lattice analysis was performed motivated by proton decay in an $SU(5)$ grand unified theory. More recently the QCDSF collaboration extended aspects of that analysis to a full examination of the 3-quark *lattice* operators including moments, [19]. Though in both cases precise measurements require continuum perturbation theory for matching. While the lattice ultimately will only require results in the Landau gauge, we choose an arbitrary gauge for internal checking purposes. For instance, in extracting the anomalous dimension of the inserted 3-quark operator, the result has to be independent of the gauge parameter in the $\overline{\text{MS}}$ scheme. This will represent a useful check here. Previously the two loop $\overline{\text{MS}}$ anomalous dimensions were computed in the Feynman gauge in [20]. The initial one loop analysis was carried out in [1, 2, 3]. In finding total agreement with the expressions of [1, 2, 3, 20] at two loops we will have another check on our results. At three loops to reduce the computation time for the large number of Feynman graphs to be evaluated, we will restrict the calculation at that order to the Feynman gauge. Though the way that part proceeds there will be an internal check over and above that provided by the renormalization group equation. It is also worth noting that the anomalous dimensions of 3-quark operators have been determined to two loops in heavy quark effective theory, [21, 22]. In that analysis one of the three quarks in the operator is regarded as having a mass significantly larger than the other two. Equally 3-quark operators have been used to estimate baryon masses using the operator product expansion, [23, 24, 25, 26, 27, 28, 29].

While this summary covers our aims it would be remiss at this stage not to mention several technical problems which have to be addressed. The 3-quark operators share a similar feature to four-fermi operators. Not only can they mix under renormalization but within dimensional regularization in $d = 4 - 2\epsilon$ spacetime dimensions, which we use throughout and where ϵ is the regularizing parameter, the operators will mix into evanescent operators. These exist in the analytically continued spacetime but are non-existent in strictly four dimensions. However, their presence within the renormalization has to be taken into account. We will use the projection technique of [30]. Coupled to this is the underlying γ^5 issue due to the fermion handedness. For automatic symbolic manipulation calculations a method was developed in [31] to accommodate γ^5 with operator renormalization. We adopt and adapt that technique for the 3-quark operators relating to the proton. At two loops the γ^5 problem does not arise in $\overline{\text{MS}}$ for the operator anomalous dimension due to the nature of the mixing matrix. At three loops the problem will be evident and needs to be treated.

Finally, we will extend a more recent two loop renormalization of a 3-quark operator which was introduced in [32]. In [32] rather than use an initial operator with the correct quantum numbers the most general 3-quark operator, devoid of any γ -matrix structure, was renormalized using dimensional regularization and an $\overline{\text{MS}}$ scheme subtraction. Clearly the seed operator

will mix into a set of 3-quark operators with γ -matrices. These arise from the internal quark propagators and quark-gluon vertices of the Feynman diagrams at each order. The claim in [32] is that from this mixing one can overcome not only γ^5 issues but also handle the evanescent operator problem in a systematic way. This is motivated by earlier work of [33]. Hence we will first check the two loop general anomalous dimension of [32] in an arbitrary linear covariant gauge and then extend the result of [32] to three loops. The former will also play the role of a subsidiary check on our proton operator $\overline{\text{MS}}$ result. Though it is worth stressing that the scheme dependent parts of the two loop results of [32] are not the same as those of [20] which is regarded as being the $\overline{\text{MS}}$ scheme. As noted in [32] they have provided a conversion factor which appears to derive from $O(\epsilon)$ corrections similar to contributions from evanescent terms. However, one advantage of the general operator of [32] is that the anomalous dimension of 3-quark operators of the spin- (j, \bar{j}) Lorentz type can be easily deduced from the general anomalous dimension. Hence we will provide the three loop value for, say, the Ioffe current [23] among other quantities.

The article is organized as follows. Section 2 is devoted to the background for the renormalization of the operators we will examine at three loops including the definition of the operator basis in d -dimensions. We provide technical details of the calculation of the Green's functions of interest in both momentum configurations in section 3. The results of the renormalization are recorded in section 4 including the finite renormalization required to treat γ^5 . The explicit form of the two loop amplitude relevant for lattice matching is presented in section 5. Section 6 is devoted to the extension of [32] to three loops where we record the anomalous dimensions of various spin operators at three loops in the scheme used in [32]. Finally, we conclude with remarks in section 7.

2 Background.

We begin by first discussing the actual operators we will renormalize. For further background we note that we have based this on the analysis of [20]. Clearly the proton involves up and down quarks and the relevant operators in the chiral limit with the correct parity are

$$\begin{aligned}\mathcal{O}_1^{udu} &= \epsilon^{IJK} \gamma^5 u^I \left((u^J)^T C d^K \right) \\ \mathcal{O}_2^{udu} &= \epsilon^{IJK} u^I \left((u^J)^T C \gamma^5 d^K \right)\end{aligned}\tag{2.1}$$

which carry one free spinor index. Here C is the charge conjugation matrix which satisfies $CC = -1$ with

$$C(\gamma^\mu)^T C = \gamma^\mu .\tag{2.2}$$

The indices I, J and K are $SU(3)$ colour indices and throughout we work in this specific Lie group. Therefore, within our calculations products of the group generators are simplified automatically with

$$T_{IJ}^a T_{KL}^a = \frac{1}{2} \left[\delta_{IL} \delta_{KJ} - \frac{1}{3} \delta_{IJ} \delta_{KL} \right]\tag{2.3}$$

where $1 \leq a \leq 8$. This means that when, for example, we make use of the quark wave function and gauge parameter anomalous dimensions as well as the QCD β -function then the usual group Casimirs, C_F , C_A and T_F , of those expressions are also evaluated at their $SU(3)$ values. Indeed we stress that the 3-quark operators we consider here have no physical meaning for colour groups other than $SU(3)$ since only for that group are they gauge invariant.

In (2.1) we have two operators of the same dimension and discrete symmetries. These will mix under renormalization. However, various linear combinations will produce the correct number of

left and right handed quarks to produce the operator which actually corresponds to the proton itself. As was discussed in [20] for practical computational purposes it is more appropriate to use a related set of operators to perform our two and three loop calculations. These operators are given by

$$\begin{aligned}\mathcal{O}_1 &= \epsilon^{IJK} \psi^I \left((\psi^J)^T C \psi^K \right) \\ \mathcal{O}_2 &= \epsilon^{IJK} \gamma^5 \psi^I \left((\psi^J)^T C \gamma^5 \psi^K \right) ,\end{aligned}\tag{2.4}$$

where we have omitted the flavour indices, and are related to the operators of (2.1) by

$$\mathcal{O}_i = \gamma^5 \mathcal{O}_i^{udu} .\tag{2.5}$$

From now on our focus will be on \mathcal{O}_i . Though we note at this point that whichever set of operators one uses they are gauge invariant for $SU(3)$ colour and so in the $\overline{\text{MS}}$ scheme, which we will use, the anomalous dimensions will be independent of the gauge parameter. This will be checked explicitly to two loops. The original computations of [20] were performed in the Feynman gauge rather than the arbitrary linear covariant gauge we use here to two loops.

For instance, as we have two operators of the same dimension and symmetries their mixing under renormalization is handled by a mixing matrix of renormalization constants and thence a matrix of anomalous dimensions. For the former we therefore have

$$\mathcal{O}_{i\text{o}} = Z_{ij} \mathcal{O}_j\tag{2.6}$$

where the subscript o denotes the bare operator. In our conventions the matrix of anomalous dimensions is defined by

$$\gamma_{ij}(a) = - \mu \frac{d}{d\mu} \ln Z_{ij}\tag{2.7}$$

where

$$\mu \frac{d}{d\mu} = \beta(a) \frac{\partial}{\partial a} + \alpha \gamma_\alpha(a, \alpha) \frac{\partial}{\partial \alpha}\tag{2.8}$$

and μ is the renormalization scale introduced to ensure that the coupling constant, g , is dimensionless in d -dimensions. Our gauge parameter is α with $\alpha = 0$ corresponding to the Landau gauge. We choose to work with the coupling constant a which is related to the gauge coupling constant and the strong force coupling constant, α_s , by

$$a = \frac{g^2}{16\pi^2} \quad , \quad a = \frac{\alpha_s}{4\pi} .\tag{2.9}$$

For mass dependent renormalization schemes $\gamma_{ij}(a)$ can depend on α but we have omitted any α dependence on the left side of (2.7) since we focus on $\overline{\text{MS}}$. For practical purposes in the extraction of the anomalous dimensions we note that (2.7) implies

$$\gamma_{ik}(a) Z_{kj} = Z_{ik} \gamma_{kj}(a) = - \mu \frac{d}{d\mu} Z_{ij}\tag{2.10}$$

and we have checked that the same mixing matrix emerges irrespective of which way the matrices are multiplied.

It should be stressed that this discussion about the structure of the mixing matrix, where i and j run over 1 and 2, is in essence the situation in four spacetime dimensions. However, as will be apparent later since we will be using dimensional regularization the mixing matrix is not finite dimensional in d -dimensions. It will have to be extended to an infinite dimensional case

at all orders in perturbation theory. This is because in d -dimensions operators will be generated through the renormalization which have no strictly four dimensional equivalent. Put another way their four dimensional equivalent is the zero operator and so such operators in d -dimensions are referred to as being evanescent. Their presence in this analysis is not solely because of the presence of γ^5 . These evanescent operators would arise if there was no mixing and one was only considering \mathcal{O}_1 . However, both the evanescent and γ^5 issues will have to be dealt with at the appropriate point. While we have noted that Z_{ij} will become infinite dimensional in d -dimensions, it will do so in a controlled way in that at each order in perturbation theory the matrix will be extended by the appearance of a new operator. In other words at each order in perturbation theory the matrix enlarges but remains finite dimensional.

To appreciate these remarks we need to focus on the treatment of γ -matrices. In d -dimensions one has to extend the basis of γ -matrices to an infinite set of matrices denoted by $\Gamma_{(n)}^{\mu_1 \dots \mu_n}$, [30, 34, 35], which are totally antisymmetric in the Lorentz indices and defined by

$$\Gamma_{(n)}^{\mu_1 \dots \mu_n} = \gamma^{[\mu_1} \dots \gamma^{\mu_n]} \quad (2.11)$$

where a factor of $1/n!$ is understood and n is an integer, $n \geq 0$. These generalized matrices span spinor space in d -dimensions and the underlying algebra necessary for loop calculations has been developed in various articles, [30, 34, 35]. For instance, the trace operation is isotropic with respect the basis since, [36, 37],

$$\text{tr} \left(\Gamma_{(m)}^{\mu_1 \dots \mu_m} \Gamma_{(n)}^{\nu_1 \dots \nu_n} \right) \propto \delta_{mn} I^{\mu_1 \dots \mu_m \nu_1 \dots \nu_n} . \quad (2.12)$$

Moreover, it is possible to write products of the original γ -matrices as a finite sum over $\Gamma_{(n)}^{\mu_1 \dots \mu_n}$. This can be achieved recursively by applying the relations, [35, 36, 37],

$$\Gamma_{(n)}^{\mu_1 \dots \mu_n} \gamma^\nu = \Gamma_{(n+1)}^{\mu_1 \dots \mu_n \nu} + \sum_{r=1}^n (-1)^{n-r} \eta^{\mu_r \nu} \Gamma_{(n-1)}^{\mu_1 \dots \mu_{r-1} \mu_{r+1} \dots \mu_n} \quad (2.13)$$

$$\gamma^\nu \Gamma_{(n)}^{\mu_1 \dots \mu_n} = \Gamma_{(n+1)}^{\nu \mu_1 \dots \mu_n} + \sum_{r=1}^n (-1)^{r-1} \eta^{\mu_r \nu} \Gamma_{(n-1)}^{\mu_1 \dots \mu_{r-1} \mu_{r+1} \dots \mu_n} \quad (2.14)$$

where $\eta_{\mu\nu}$ is the spacetime metric tensor. For reference when one restricts the $\Gamma_{(n)}$ -matrices to four dimensions we have

$$\begin{aligned} \Gamma_{(2)}^{\mu\nu} \Big|_{d=4} &= \sigma^{\mu\nu} \quad , \quad \Gamma_{(4)}^{\mu\nu\sigma\rho} \Big|_{d=4} = \epsilon^{\mu\nu\sigma\rho} \gamma^5 \\ \Gamma_{(n)}^{\mu_1 \dots \mu_n} \Big|_{d=4} &= 0 \quad \text{for } n \geq 5 \end{aligned} \quad (2.15)$$

where $\epsilon^{\mu\nu\sigma\rho}$ is the four dimensional totally antisymmetric pseudotensor. As a note we mention that the γ^5 matrix which exists in strictly four dimensions and defines chirality has absolutely no connection whatsoever with $\Gamma_{(5)}^{\mu_1 \dots \mu_5}$ in d -dimensions. The former object exists only in four dimensions and Larin's procedure, [31], which we use to handle γ^5 within dimensional regularization will be detailed later for the present computation. As a notational comment we will use γ^μ and $\Gamma_{(1)}^\mu$ synonymously in d -dimensions since in (2.14) as the former is less clumsy but regard $\sigma^{\mu\nu}$ as the purely four dimensional object.

There are two sources of these generalized γ -matrices within the computations. The first is the simplest and alluded to already and that is that the product of γ -matrices which remain in the calculation of either Green's function can be written in the general basis. This leads to the second source which is the generation of operators into which the seed operator, (2.4), of the

Green's function mixes under renormalization. In d -dimensions given that we have to use the generalized basis we then have to use the generalization of the four dimensional operators to the same basis. Therefore we define the new d -dimensional operators

$$\begin{aligned}\mathcal{O}_{(n)} &= \epsilon^{IJK} \left(\Gamma_{(n)}^{\mu_1 \dots \mu_n} \psi^I \right) \left((\psi^J)^T C \Gamma_{(n) \mu_1 \dots \mu_n} \psi^K \right) \quad , \quad \text{for } n \neq 4 \\ \mathcal{O}_{(4)} &= \frac{1}{24} \epsilon^{IJK} \left(\Gamma_{(4)}^{\mu_1 \mu_2 \mu_3 \mu_4} \psi^I \right) \left((\psi^J)^T C \Gamma_{(4) \mu_1 \mu_2 \mu_3 \mu_4} \psi^K \right) \quad , \quad \text{for } n = 4\end{aligned}\quad (2.16)$$

where the factor for $\mathcal{O}_{(4)}$ is to ensure that there is a direct mapping in the four dimensional limit to the original operators. This follows from (2.15) and the fact that in $\epsilon_{\mu\nu\sigma\rho}\epsilon^{\mu\nu\sigma\rho} = 24$. Hence,

$$\mathcal{O}_{(4)} = \mathcal{O}_2 + \mathcal{O}(\epsilon) \quad (2.17)$$

in the limit to four dimensions. As a mnemonic we note that

$$\mathcal{O}_n = \mathcal{O}_{(4n-4)} \quad (2.18)$$

for $n \geq 1$ and

$$\begin{aligned}\mathcal{O}_1|_{d=4} &= \epsilon^{IJK} \psi^I \left((\psi^J)^T C \psi^K \right) \\ \mathcal{O}_2|_{d=4} &= \epsilon^{IJK} (\gamma^5 \psi^I) \left((\psi^J)^T C \gamma^5 \psi^K \right) \\ \mathcal{O}_n|_{d=4} &= 0 \quad \text{for } n \geq 3.\end{aligned}\quad (2.19)$$

Thus at the outset we are forced to consider a more general renormalization from which the anomalous dimensions of the original operators will emerge as a corollary. So within all our computations we will write the Green's functions in terms of $\Gamma_{(n)}$ -matrices. For the finite parts, which is ultimately required for lattice matching, the Lorentz indices on these matrices can be contracted with the external momenta. Though such objects will not be associated with a divergence in ϵ since this would violate the renormalizability of the operators. Before restricting the d -dimensional Green's function there are more channels. This is one of the reasons why we have not taken the projection approach which was the main tool for mesonic operators, [14, 15]. For instance, one has to have knowledge of the full basis to say three loops for (2.26) and then construct the projection tensor to isolate all the possible spinor channels. Within a symbolic manipulation approach this would significantly reduce run times due, in part, to having to internally manipulate products of $\Gamma_{(n)}$ -matrices. Instead we have constructed the relations between the products of $\Gamma_{(m)}^{\mu_1 \dots \mu_m}$ and $\Gamma_{(n)}^{\nu_1 \dots \nu_n}$ for various values of m and n which occur. These have been encoded within a module in the symbolic manipulation language, FORM, [38] which we use throughout. Such a product can be written in terms of $\Gamma_{(p)}^{\sigma_1 \dots \sigma_p}$ where $|m - n| \leq p \leq (m + n)$ and $\sigma_i \in \{\mu_1, \dots, \mu_m, \nu_1, \dots, \nu_n\}$. The indices not used from this set in $\Gamma_{(p)}^{\sigma_1 \dots \sigma_p}$ appear in the $\eta^{\mu\nu}$ tensors which are required to keep the total number of free indices of each term as $(m + n)$.

Given the appearance now of the full basis of operators $\mathcal{O}_{(n)}$ in d -dimensions we will determine the associated renormalization constant matrix Z_{ij} . This matrix will increase in size at each loop order but from it we will determine what we will refer to as the naive anomalous dimension matrix, $\tilde{\gamma}_{ij}(a)$. It would ordinarily correspond to the correct four dimensional anomalous dimensions but in using these generalized operators we have ignored the problem of γ^5 as well as the effect of the evanescent operators. The latter affect the structure of the four dimensional anomalous dimensions even though the evanescent operators are non-existent in four spacetime dimensions. To account for this we note the formalism developed in [30] which appends to the naive anomalous dimensions extra contributions which derive from the evanescent parts. We make minimal comment on the technique here since it transpires that the effect they

have on the proton operator renormalization will not occur until *four* loops. Though if there were contributions these together with the naive anomalous dimensions would contribute to the correct four dimensional result. We say contribute as one has also to deal with the absence of chiral symmetry in d -dimensions which has been ignored with the choice of $\mathcal{O}_{(4)}$. Including the evanescent effects with the naive anomalous dimensions for the present case would not produce a result consistent with chiral symmetry in strictly four dimensions. The procedure we have chosen to do this is based on Larin's method, [31], which was developed for flavour non-singlet and singlet quark currents as well as the chiral anomaly. We will discuss the technical aspects of the calculation for our case later but in essence one needs to append a *finite* renormalization constant to the naive renormalization constant. It is chosen in such a way that in strictly four dimensions the anti-commutativity of γ^5 with γ^μ is restored. This aspect is treated *after* the contributions from the evanescent operators have been included. The criterion for defining the condition will be similar for our operators \mathcal{O}_i , $i = 1$ and 2 , and like [31] will be derived from knowledge of the finite part of the Green's functions with the operators inserted. Though the finite renormalization can also be derived from the difference in anomalous dimensions. However, in either case unlike [31] it will be a matrix of finite renormalization constants leading to an additional matrix of anomalous dimensions, $\gamma_{5,ij}(a)$. Thus the correct four dimensional anomalous dimension matrix for (2.1) or (2.4) will formally be

$$\gamma_{ij}(a) = \tilde{\gamma}_{ij}(a) + \gamma_{5,ij}(a) \quad (2.20)$$

where we exclude any evanescent part for this case. Here

$$\gamma_{5,ij}(a) = -\mu \frac{d}{d\mu} Z_{ij}^5 \quad (2.21)$$

and Z_{ij}^5 is the finite renormalization constant matrix. If the original seed operators had included additional γ -matrices, with or without free Lorentz indices such as those of [24, 39], then the evanescent operator contribution could occur at the three loop order we are interested in here. In writing $\gamma_{5,ij}(a)$ we are assuming that the finite renormalization is independent of the gauge parameter. In the cases examined by Larin in [31] and in a more recent analysis of diquark operators, [40], the finite renormalization did not depend on α which is what we found here. Indeed it is worth noting that as this additional piece corresponds to a finite renormalization, for the *three* loop mixing matrix in four dimensions one only requires Z_{ij}^5 to *two* loops because of the presence of $\beta(a)$ in (2.21).

We close this section by summarizing the two calculations we carry out. First, in each case a 3-quark operator is inserted at zero momentum into a quark 3-point function. For illustration if for the moment we denote this generic operator by \mathcal{O} then we will calculate

$$\langle \psi_\alpha(p) \psi_\beta(q) \psi_\gamma(r) \mathcal{O}_\delta(0) \rangle \Big|_{p^2=q^2=r^2=-\mu^2} \quad (2.22)$$

to two loops to the finite part for arbitrary gauge parameter α in the $\overline{\text{MS}}$ scheme. This is to assist with lattice matching to the same quantity in the Landau gauge. For this case there are 3 one loop and 40 two loop Feynman graphs to compute and we will call this the symmetric setup. In (2.22) momentum conservation implies

$$r = -p - q \quad (2.23)$$

and we use a symmetric subtraction point for the external legs

$$p^2 = q^2 = r^2 = -\mu^2 \quad (2.24)$$

which implies

$$pq = \frac{1}{2}\mu^2. \quad (2.25)$$

We have chosen the subtraction point to be $(-\mu^2)$ so as to omit logarithms in the finite part of the Green's function. In order to extract the three loop mixing matrix of anomalous dimensions we have to consider a different momentum configuration so that the MINCER algorithm, [7], can be applied. We will refer to this as the MINCER setup. In this case the Green's function is

$$\langle \psi_\alpha(p) \psi_\beta(-p) \psi_\gamma(0) \mathcal{O}_\delta(0) \rangle \Big|_{p^2=-\mu^2} \quad (2.26)$$

and there are 784 three loop diagrams to determine in addition to those noted earlier at lower order. In both situations, (2.22) and (2.26), the Feynman diagrams are generated automatically using the QGRAF package, [41], before being converted into FORM, [38], input notation. The latter is the symbolic manipulation language we use and the full automatic computation is written in terms of it. Though the compilation of the expressions for each Feynman diagram for both setups were run with the threaded version, TFORM, [42]. The nullification of an external quark leg momentum in effect produces a 2-point function which MINCER requires.

3 Computational technicalities.

We devote this section to the various technical aspects of the computation. In order to evaluate the Green's function for the full momentum or symmetric configuration we use the Laporta algorithm, [43]. Each of the one and two loop Feynman graphs we have to evaluate involve strings of γ -matrices which have external and internal momenta embedded within them. We have proceeded by writing all integrals in terms of scalar integrals. By this we mean integrals where there are at most scalar products of the momenta and the strings of γ -matrices have contractions with only external momenta. In other words these only play a passive role in the subsequent evaluation and are written in terms of the generalized $\Gamma_{(n)}$ -matrices. To achieve this we make a projection of the tensor integrals onto a basis of tensors built from $\eta_{\mu\nu}$, p_μ and q_μ . For the tensor reduction we use we have at most a rank 5 tensor built from the two internal loop momenta at two loops. This is due to the fact that we are computing for an arbitrary linear covariant gauge. A Feynman gauge computation would be more compact but would have limited applicability for the lattice. Once the scalarized integrals have been determined these are rewritten purely in terms of the propagators of the graph and any additional propagators which are not part of the topology, [43]. These latter propagators are required for irreducible scalar products but are chosen in such a way as to cover all possible scalar products of the internal momenta with themselves and the external momenta. In this form one applies the Laporta reduction. This is an algorithm which systematically constructs all the integration by parts relations and optionally the Lorentz identities between all the integrals which are needed. From this tower of relations it is possible to algebraically relate all the scalar integrals to a base set of master integrals. These are evaluated by explicit integration and thus the evaluation of the Feynman graph is complete.

In describing the general procedure we note that for practical purposes one has to use computers to implement the Laporta algorithm. We have used REDUZE, [44], which uses the GINAC computer algebra system, [45], and is written in C++. For the one and two loop graphs we need to evaluate it transpires that there are three basic topologies. There is one at one loop and two at two loops. For the latter one is the ladder graph and the other is the non-planar two loop 3-point graph. These and their extension to include one other propagator are sufficient to cover all possible irreducible tensor integrals. Thus using the REDUZE package we have created

a database of relations covering all possible levels of scalar integrals which can arise. The ones which are explicitly required are extracted and converted into a FORM module which is called at the appropriate point of the automatic computation. At the end the explicit expressions for the master integrals are substituted. The ones we use are summarized in [13] but were evaluated in various articles, [46, 47, 48, 49], using a variety of techniques.

For the MINCER situation, (2.26), the momentum configuration with one nullified external quark leg could potentially introduce spurious infrared infinities which would need infrared rearrangement. However, this does not arise. This would be the case if the Feynman integral produced propagators such as $1/(k^2)^2$, where k is an internal loop momentum, which are infrared singular. These are absent because the quark propagator retains \not{k} in the numerator or the triple gluon or ghost vertices carry a momentum to lift the potential infrared singularity. By contrast to (2.22) we take a more general operator in order to compute the anomalous dimension. In particular we seed the Green's function with

$$\mathcal{O} = \epsilon^{IJK} \psi_\alpha^I \psi_\beta^J \psi_\gamma^K \quad (3.1)$$

which is not decorated with γ -matrices and α, β and γ are spinor indices. There are several reasons for doing this. One is a practical one to do with the size of the three loop calculation. There are 784 three loop diagrams to determine and thus to keep computer run times to a minimum it transpires that it is more efficient to evaluate the diagrams with (3.1) and then introduce the γ -matrix structure appropriate to each original operator of (2.16) when summing the diagrams. In addition it also allows one to quickly construct that part of the mixing matrix relating to the evanescent operators which are generated rather than have to repeat a full run. Indeed in that case the calculation would necessarily be slower as it could involve $\Gamma_{(n)}^{\mu_1 \dots \mu_n}$ for values of n up to 20 for each of the three loop graphs. Another reason for proceeding with (3.1) is that we can extend the recent calculation of [32] at the *same* time.

However, in choosing to calculate in this more general way there are several technical issues to be overcome which are rooted in the MINCER algorithm. It computes massless scalar 2-point functions to three loops in dimensional regularization. With the 3-quark operator zero momentum insertion in a quark 3-point function the nullification of an external quark leg momentum ensures we have in effect a 2-point function immediately. To obtain scalar integrals within the symbolic representation of each Feynman diagram we strip off the γ -matrix structure from the numerator as for (2.22). An alternative approach would be to project out the Lorentz structure but this is too cumbersome especially as one has to have prior knowledge of the full structure of the Green's function at each loop order. Removing the γ -matrices instead leaves each diagram as a sum of Lorentz tensor integrals. Given that we are in the chiral limit these integrals will be of even rank. In the Feynman gauge they will be rank $(2l)$ where l is the loop order. By contrast for a general linear covariant gauge these integrals will be at most rank $(4l)$. In addition the MINCER algorithm, [7], codes the internal loop momenta for each of the MINCER topologies in terms of its own labelling. In particular each line of a topology is assigned an internal momentum label p_i . The conservation of energy momentum at each vertex is then encoded within the integration routine for that topology. In other words there are no MINCER labels such as $(p_1 - p_2)$ or $(Q - p_3 - p_6)$ where Q is the external momentum of the 2-point function. Thus all the tensor integrals involve products of internal momentum vectors p_i where $1 \leq i \leq 8$ for three loops. For lower loop orders there are fewer internal momentum labels. Therefore, the problem of evaluating each graph of the Green's function requires converting these even rank tensors into scalar integrals, which are straightforward to compute in MINCER, and Lorentz tensors built from $\eta_{\mu\nu}$ and Q_μ . The procedure for this is straightforward. Using only knowledge of the rank we have written down the most general tensor basis for each rank and then determined the scalar integral amplitude by the method of projection for each tensor. For instance, for rank

2, 4 and 6 there are respectively 2, 10 and 76 tensors. Using FORM we have constructed the decomposition for an arbitrary numerator which is straightforward. For rank 8 the projection matrix would be 764×764 . Rather than use a decomposition into $\eta_{\mu\nu}$ and Q_μ we chose the transverse and longitudinal projection tensors

$$P_{\mu\nu}(Q) = \eta_{\mu\nu} - \frac{Q_\mu Q_\nu}{Q^2} \quad , \quad L_{\mu\nu}(Q) = \frac{Q_\mu Q_\nu}{Q^2} \quad (3.2)$$

which satisfy the simple properties

$$\begin{aligned} P_{\mu\nu}(Q) + L_{\mu\nu}(Q) &= \eta_{\mu\nu} \quad , \quad P_{\mu\nu}(Q)P^{\nu\sigma}(Q) = P_\mu{}^\sigma(Q) \\ P_{\mu\nu}(Q)L^{\nu\sigma}(Q) &= 0 \quad , \quad L_{\mu\nu}(Q)L^{\nu\sigma}(Q) = L_\mu{}^\sigma(Q) . \end{aligned} \quad (3.3)$$

The benefit of this is that the full matrix becomes block diagonal with the largest submatrix being 105×105 . Once the general decomposition has been derived it is encoded as an integration module in FORM at the appropriate point in the overall algorithm. With rank 8 decomposition the anomalous dimension matrix can be determined to two loops in an arbitrary linear covariant gauge. At this point we note that since the operators we consider are gauge invariant this allows us to check that in $\overline{\text{MS}}$ the two loop anomalous dimensions are independent of the gauge parameter. This is a strong check on the derivation of the FORM module encoding the mapping of the Lorentz tensor integrals to the scalar integrals. Indeed the two loop result of [20] was performed only in the Feynman gauge. For three loops one could in principle extend the construction to rank 12. Instead we have taken a different approach and chosen to calculate the three loop diagrams solely in the Feynman gauge. In this case this only requires the decomposition up to rank 6 which has already been tested for the arbitrary gauge calculation at two loops. Moreover, in the Feynman gauge the diagrams can be evaluated more quickly. The independent check on the three loop anomalous dimensions will be consistency with the renormalization group equation as the double and triple poles in ϵ are already fixed from the one and two loop renormalization constants. We believe this calculational approach from the point of view of exploiting the properties of the gluon propagator in various gauges is the most efficient to deduce the anomalous dimensions. However, the finite part of this Green's function at three loops is not useful for lattice matching as the Landau gauge results will not be determined. Indeed it is not clear whether they would be meaningful anyway in this instance as the nullification of the momentum of an external quark leg could be problematic from the point of view of infrared divergences in the chiral limit.

4 Three loop anomalous dimensions.

While we have split the discussion on the technical aspects of each setup we collect the results of our renormalization in one section. This is because for both calculations we obtain the same results to two loops for an arbitrary linear covariant gauge. As indicated in the MINCER discussion we have reasonable internal checks on the three loop part of the anomalous dimensions from the renormalization group equations. First, we record the naive anomalous dimensions for the seed operators $\mathcal{O}_{(0)}$ and $\mathcal{O}_{(4)}$ where the labels 1 and 2 refer to these operators respectively. To three loops in $\overline{\text{MS}}$ we have

$$\begin{aligned} \tilde{\gamma}_{11}(a) &= -2a - [2N_f + 51] \frac{a^2}{9} \\ &\quad + [260N_f^2 + [4320\zeta(3) - 4656]N_f + 1296\zeta(3) + 23481] \frac{a^3}{162} + O(a^4) \end{aligned}$$

$$\begin{aligned}
\tilde{\gamma}_{12}(a) &= \frac{10}{3}a^2 + [216\zeta(3) - 153 - 14N_f] \frac{a^3}{27} + O(a^4) \\
\tilde{\gamma}_{21}(a) &= \frac{10}{3}a^2 + 4[18\zeta(3) + 331 - 22N_f] \frac{a^3}{9} + O(a^4) \\
\tilde{\gamma}_{22}(a) &= -2a - [2N_f + 51] \frac{a^2}{9} \\
&\quad + [260N_f^2 + [4320\zeta(3) + 1344]N_f + 1296\zeta(3) - 75519] \frac{a^3}{162} + O(a^4) \quad (4.1)
\end{aligned}$$

where $\zeta(z)$ is the Riemann ζ -function. The two loop parts are calculated for arbitrary α and agree with the full two loop mixing matrix of [20]. This is because at this order the γ^5 problem can be ignored as was noted in [20] and hence the two loop naive anomalous dimensions are sufficient to determine the proton operator wave function renormalization. As we generate $\mathcal{O}_{(8)}$ or \mathcal{O}_3 we have to include the next row and column of $\tilde{\gamma}_{ij}(a)$ to the appropriate orders in a . Thus we have

$$\begin{aligned}
\tilde{\gamma}_{33}(a) &= -2a - [2N_f - 19869] \frac{a^2}{9} + O(a^3) \\
\tilde{\gamma}_{23}(a) &= \frac{5}{864}a^2 + [4824\zeta(3) - 2745 - 14N_f] \frac{a^3}{15552} + O(a^4) \\
\tilde{\gamma}_{31}(a) &= \tilde{\gamma}_{32}(a) = O(a^3) \\
\tilde{\gamma}_{13}(a) &= O(a^4) . \quad (4.2)
\end{aligned}$$

The key terms in this are those in the upper triangle. In the formalism of [30] $\tilde{\gamma}_{31}(a)$ and $\tilde{\gamma}_{32}(a)$ correspond to the generalized β -functions used in the evanescent projection formalism to extract the effect the evanescent operators have on the strictly four dimensional anomalous dimensions, before we restore chiral symmetry and the anti-commutativity of γ^5 . As these two anomalous dimensions are zero to $O(a^3)$ then there can be no evanescent operator affect until *four* loops in the full four dimensional anomalous dimensions.

In order to complete the computation we require the finite renormalization constants which restore the anti-commutativity of γ^5 in strictly four dimensions. As background it is worthwhile recalling the Larin procedure, [31], for extracting the finite renormalization constant for the flavour non-singlet pseudo-scalar quark current, $\mathcal{O}_{m_5} = \bar{\psi}\gamma^5\psi$. Here we assume the operator is inserted in a quark 2-point function at zero momentum. With a naive anti-commuting γ^5 in d -dimensions one would simply anti-commute the γ^5 out of all the diagrams. However, retaining it inside diagrams and using the generalized $\Gamma_{(n)}$ -matrices to compute the Green's function one obtains a finite Green's function in the four dimensional limit after the renormalization constants have been determined in the $\overline{\text{MS}}$ scheme. The value for this should be equivalent to that calculated with the naive anti-commuting γ^5 in d -dimensions. However, it is not the same and therefore to remove the discrepancy one defines a finite renormalization constant, [31]. If we define this as Z^5 for the pseudo-mass operator then more specifically the condition is defined by

$$\gamma^5 \langle \psi(p) \mathcal{O}_m(0) \bar{\psi}(-p) \rangle|_{d=4} = Z^5 \langle \psi(p) \mathcal{O}_{m_5}(0) \bar{\psi}(-p) \rangle|_{d=4} \quad (4.3)$$

where $\mathcal{O}_m = \bar{\psi}\psi$ is the quark mass operator. While this is the situation for a single operator with no mixing the renormalization of the 3-quark operator is complicated by the mixing. In this respect it is notationally difficult to extend the definition of (4.3) to the matrix case. At a formal level the definition can be represented by

$$\langle \mathcal{O}_1 \rangle = Z^5 \odot (\gamma^5 \otimes \gamma^5) \langle \mathcal{O}_2 \rangle \quad (4.4)$$

where the tensor product of the γ^5 matrices acts on the appropriate spinor indices of the Green's function. Also Z^5 represents the finite renormalization matrix and its effect within the finite

part of the Green's function which is the meaning of the \odot multiplication. While this is the normal Larin procedure where the naive $\overline{\text{MS}}$ anomalous dimensions are computed first and then the finite renormalization, in practical terms it is quicker to do both processes together. In other words we absorb a certain finite part into our $\overline{\text{MS}}$ renormalization constants. The condition for this is that the finite parts of both Green's functions after renormalization in four dimensions are equivalent up to multiplying by $\gamma^5 \otimes \gamma^5$. However, to assist others who wish to reproduce the results we will *present* them as in the two stage Larin method. Thus in addition to the naive anomalous dimensions we have extracted the anomalous dimensions associated with the finite γ^5 renormalization matrix. These are

$$\begin{aligned}\gamma_{5,11}(a) &= \gamma_{5,12}(a) = O(a^4) \\ \gamma_{5,21}(a) &= 125 [2N_f - 33] \frac{a^3}{27} + O(a^4) \\ \gamma_{5,22}(a) &= -500 [2N_f - 33] \frac{a^3}{27} + O(a^4)\end{aligned}\tag{4.5}$$

where the linear factor in N_f derives from the one loop β -function for $SU(3)$. In defining the finite renormalization condition (4.4) we have not included the momentum configuration of the external quark legs. This is because (4.5) has been derived for both setups. The emergence of the same finite renormalization from both calculations is an independent check. Moreover both computations were performed in an arbitrary linear covariant gauge. However, as is evident from (4.5) the result is independent of α . For all the quark bilinear current operators considered in [31] and the diquark operators examined in [40] which contained γ^5 , the associated finite renormalization constant was also independent of the gauge parameter. Indeed this follows from a simple observation that the finite renormalization constant in the bilinear current cases is related via renormalization group arguments to the *naive* anomalous dimensions of the particular operators involved in the defining condition. In the case of our earlier example this would be the naive anomalous dimensions of \mathcal{O}_m and \mathcal{O}_{m_5} . As the operators are gauge invariant and being renormalized in a mass independent scheme then as the operator anomalous dimensions are independent of the gauge parameter then so to is the finite renormalization constant. By contrast if the original operator renormalization had been performed in a mass dependent scheme then not only would the operator anomalous dimensions be gauge dependent but the finite renormalization constant would too. This is of course barring accidental cancellations of the gauge parameter which is in principle possible. For the case of the operators \mathcal{O}_1 and \mathcal{O}_2 a similar argument can be established. Though due to the mixing it derives from within the formal renormalization condition (4.4) and the specific way we have defined the finite renormalization constant within the explicit calculation of the Green's function of each operator in the one stage method of applying the Larin technique which we used here. To summarize the upshot of that analysis translates into the equations

$$\begin{aligned}\tilde{\gamma}_{11}(a) - \tilde{\gamma}_{22}(a) &= \mu \frac{d}{d\mu} \ln Z_{22}^5 \\ \tilde{\gamma}_{12}(a) - \tilde{\gamma}_{21}(a) &= \mu \frac{d}{d\mu} \ln Z_{21}^5.\end{aligned}\tag{4.6}$$

These equations are analogous to the situation in the quark current example. As each of the naive operator anomalous dimensions are gauge independent, since the operators themselves are gauge invariant, and we are in the $\overline{\text{MS}}$ scheme, which is a mass independent scheme, then the two finite renormalization constants are automatically also gauge independent. Indeed from the explicit expressions for the naive three loop anomalous dimensions it is straightforward to check that (4.5) are consistent with these general expressions. This is a reassuring check since we derived (4.5) from the finite part of (4.4) which was calculated in an arbitrary linear

covariant gauge in both calculational setups to two loops. It was only the three loop MINCER calculational which was carried out in the Feynman gauge. Therefore, there is consistency with that calculation too.

Equipped with this finite renormalization we can now determine the full four dimensional mixing matrix using the naive anomalous dimensions. We find to three loops that

$$\begin{aligned}
\gamma_{11}(a) &= \gamma_{22}(a) = -2a - [2N_f + 51] \frac{a^2}{9} \\
&\quad + [260N_f^2 + [4320\zeta(3) - 4656]N_f + 1296\zeta(3) + 23481] \frac{a^3}{162} \\
&\quad + O(a^4) \\
\gamma_{12}(a) &= \gamma_{21}(a) = \frac{10}{3}a^2 + [216\zeta(3) - 153 - 14N_f] \frac{a^3}{27} + O(a^4) .
\end{aligned} \tag{4.7}$$

The effect of the finite renormalization has been to restore the symmetry of the mixing matrix so that the diagonal entries are equal and the off-diagonal are the same but different to the other two. This structure was present at one and two loops, [1, 2, 3, 20]. However, for the actual proton anomalous dimension we need to have the correct handedness of the up and down quarks which requires the eigen-anomalous dimensions which are

$$\begin{aligned}
\gamma_+(a) &= -2a - [2N_f + 21] \frac{a^2}{9} + [260N_f^2 + [4320\zeta(3) - 4740]N_f + 2592\zeta(3) + 22563] \frac{a^3}{162} \\
&\quad + O(a^4) \\
\gamma_-(a) &= -2a - [2N_f + 81] \frac{a^2}{9} + [260N_f^2 + [4320\zeta(3) - 4572]N_f + 24399] \frac{a^3}{162} \\
&\quad + O(a^4)
\end{aligned} \tag{4.8}$$

where

$$\gamma_+(a) = \gamma_{11}(a) + \gamma_{12}(a) , \quad \gamma_-(a) = \gamma_{11}(a) - \gamma_{12}(a) . \tag{4.9}$$

It is the latter, $\gamma_-(a)$, which corresponds to the proton. Numerically we have

$$\begin{aligned}
\gamma_+(a) &= -2.0000000a - [0.2222222N_f + 2.3333333] a^2 \\
&\quad + [1.6049383N_f^2 + 2.7955915N_f + 158.5106882] a^3 + O(a^4) \\
\gamma_-(a) &= -2.0000000a - [0.2222222N_f + 9.0000000] a^2 \\
&\quad + [1.6049383N_f^2 + 3.8326285N_f + 150.6111111] a^3 + O(a^4) .
\end{aligned} \tag{4.10}$$

Having established the anomalous dimensions we can construct the renormalization group invariant current, j_- , using the same notation as [20]. It is defined in the conventional way by

$$j_- = \exp \left(\int^a \frac{\gamma_-(z)}{\beta(z)} dz \right) \bar{j}_-(\mu) \tag{4.11}$$

and satisfies

$$\mu \frac{dj_-}{d\mu} = 0 . \tag{4.12}$$

Solving

$$j_- \equiv \gamma_{j_-}(a) \bar{j}_-(\mu) \tag{4.13}$$

explicitly we formally have

$$\begin{aligned}
\gamma_{j_-}(a) &= a^{\gamma/\beta_1} \left[1 + \left[\frac{\gamma_2}{\beta_1} - \frac{\gamma_1\beta_2}{\beta_1^2} \right] a \right. \\
&\quad + \left[\frac{\gamma_3}{\beta_1} - \frac{\gamma_2\beta_2}{\beta_1^2} - \frac{\gamma_1\beta_3}{\beta_1^2} + \frac{\gamma_2^2}{\beta_1^2} + \frac{\gamma_1\beta_2^2}{\beta_1^3} - \frac{2\gamma_1\gamma_2\beta_2}{\beta_1^3} + \frac{\gamma_1^2\beta_2^2}{\beta_1^4} \right] \frac{a^2}{2} \\
&\quad \left. + O(a^3) \right]
\end{aligned} \tag{4.14}$$

where

$$\begin{aligned}\gamma_-(a) &= \gamma_1 a + \gamma_2 a^2 + \gamma_3 a^3 + O(a^4) \\ \beta(a) &= \beta_1 a^2 + \beta_2 a^3 + \beta_3 a^4 + O(a^5) .\end{aligned}\tag{4.15}$$

This produces

$$\begin{aligned}\gamma_{j-}(a) &= \left[1 - [4N_f^2 - 588N_f + 2835] \frac{a}{3[2N_f - 33]^2} \right. \\ &\quad + [2080N_f^5 + [34560\zeta(3) - 157368]N_f^4 + [4596912 - 1710720\zeta(3)]N_f^3 \\ &\quad + [28226880\zeta(3) - 70113330]N_f^2 + [580876920 - 155247840\zeta(3)]N_f \\ &\quad \left. - 1825381251] \frac{a^2}{108[2N_f - 33]^4} + O(a^3) \right] a^{6/[33-2N_f]} .\end{aligned}\tag{4.16}$$

For three flavours this gives

$$\gamma_{j-}(a)|_{N_f=3} = \left[1 - \frac{41}{81}a - \frac{[116640\zeta(3) + 275215]}{26244}a^2 + O(a^3) \right] a^{2/9}\tag{4.17}$$

or

$$\gamma_{j-}(a)|_{N_f=3} = [1 - 0.5061728a - 15.8292531a^2 + O(a^3)] a^{2/9}\tag{4.18}$$

numerically. To gauge the effects of the two loop correction we can compare the numerical value of $\gamma_{j-}(a)|_{N_f=3}$ at one loop with that at two loops for $\alpha_s = 0.1$. We find that the two loop correction modifies the one loop value by around 0.1%.

5 Amplitudes

In this section we record the explicit values of the Green's function (2.22) to two loops in an arbitrary linear covariant gauge in the $\overline{\text{MS}}$ scheme. This represents one of the main results of the article as it will be of use for lattice matching. We have*

$$\begin{aligned}\langle \psi_\alpha(p) \psi_\beta(q) \psi_\gamma(-p-q) \mathcal{O}_{1\delta}(0) \rangle \Big|_{\text{symm}} &= \left[1 + \frac{2}{9} [2\pi^2\alpha + 15\alpha + 2\pi^2 + 15 \right. \\ &\quad \left. - 3\psi'(\tfrac{1}{3}) - 3\psi'(\tfrac{1}{3})\alpha] a \right. \\ &\quad + [[2160\psi'(\tfrac{1}{3}) - 1440\pi^2 - 19008] N_f + 1440\psi'(\tfrac{1}{3})^2 \\ &\quad - [8262\alpha^2 + 23652\alpha + 122742] \psi'(\tfrac{1}{3}) \\ &\quad - 1920\psi'(\tfrac{1}{3})\pi^2 + [711\alpha + 207] \psi'''(\tfrac{1}{3}) \\ &\quad + [136080\alpha - 392688] s_2(\tfrac{\pi}{6}) \\ &\quad + [785376 - 272160\alpha] s_2(\tfrac{\pi}{2}) \\ &\quad + [654480 - 226800\alpha] s_3(\tfrac{\pi}{6}) \\ &\quad + [181440\alpha - 523584] s_3(\tfrac{\pi}{2}) + [88 - 1896\alpha] \pi^4 \\ &\quad + [5508\alpha^2 + 15768\alpha + 81828] \pi^2 \\ &\quad - [18468 + 12636\alpha] \Sigma - [73224\alpha + 23976] \zeta(3) \\ &\quad \left. + 188244\alpha + 403461 + [2929 - 1015\alpha] \frac{\pi^3}{\sqrt{3}} \right]\end{aligned}$$

*The full analytic form of the amplitude for an arbitrary gauge and the anomalous dimensions, have been included in an attached electronic data file.

$$\begin{aligned}
& + [32724 - 11340\alpha] \frac{\ln(3)\pi}{\sqrt{3}} \\
& + [945\alpha - 2727] \frac{\ln^2(3)\pi}{\sqrt{3}} \left] \frac{a^2}{2916} \right] I_{\alpha\delta} \otimes I_{\beta\gamma} \\
& + 5 \left[[648\zeta(3) - 1323 - 540\psi'(\tfrac{1}{3})\pi^2 - 3888s_2(\tfrac{\pi}{6}) \right. \\
& \quad + 7776s_2(\tfrac{\pi}{2}) + 6480s_3(\tfrac{\pi}{6}) - 5184s_3(\tfrac{\pi}{2}) \\
& \quad + 360\pi^2 + 29\frac{\pi^3}{\sqrt{3}} + 324\frac{\ln(3)\pi}{\sqrt{3}} \\
& \quad \left. - 27\frac{\ln^2(3)\pi}{\sqrt{3}} \right] \frac{a^2}{486} \right] \gamma_{\alpha\delta}^5 \otimes \gamma_{\beta\gamma}^5 \\
& + \left[4 \left[3\psi'(\tfrac{1}{3})\alpha + 3\psi'(\tfrac{1}{3}) - 2\pi^2\alpha - 2\pi^2 \right] \frac{a}{81} \right. \\
& \quad + \left[[1056\pi^2 - 1584\psi'(\tfrac{1}{3})] N_f \right. \\
& \quad - [333\alpha + 513] \psi'''(\tfrac{1}{3}) \\
& \quad + [2322\alpha^2 - 1188\alpha - 1350] \psi'(\tfrac{1}{3}) \\
& \quad - [104976\alpha + 295488] s_2(\tfrac{\pi}{6}) \\
& \quad + [209952\alpha + 590976] s_2(\tfrac{\pi}{2}) \\
& \quad + [174960\alpha + 492480] s_3(\tfrac{\pi}{6}) \\
& \quad - [139968\alpha + 393984] s_3(\tfrac{\pi}{2}) \\
& \quad + [888\alpha + 1368] \pi^4 \\
& \quad + [900 + 792\alpha - 1548\alpha^2] \pi^2 \\
& \quad + [4212\alpha + 4212] \Sigma + [15552\alpha - 5184] \zeta(3) \\
& \quad + [2204 + 783\alpha] \frac{\pi^3}{\sqrt{3}} \\
& \quad + [24624 + 8748\alpha] \frac{\ln(3)\pi}{\sqrt{3}} \\
& \quad \left. - [729\alpha + 2052] \frac{\ln^2(3)\pi}{\sqrt{3}} \right] \frac{a^2}{4374} \right] \frac{I_{\alpha\delta} \otimes \sigma_{\beta\gamma}^{pq}}{\mu^2} \\
& + 5 \left[[648\zeta(3) + 216\psi'(\tfrac{1}{3})\pi^2 - 18\psi'''(\tfrac{1}{3}) \right. \\
& \quad - 3888s_2(\tfrac{\pi}{6}) + 7776s_2(\tfrac{\pi}{2}) + 6480s_3(\tfrac{\pi}{6}) \\
& \quad - 5184s_3(\tfrac{\pi}{2}) + 48\pi^4 - 144\pi^2 + 29\frac{\pi^3}{\sqrt{3}} \\
& \quad + 324\frac{\ln(3)\pi}{\sqrt{3}} \\
& \quad \left. - 27\frac{\ln^2(3)\pi}{\sqrt{3}} \right] \frac{a^2}{4374} \right] \frac{\gamma_{\alpha\delta}^5 \otimes (\gamma^5 \sigma^{pq})_{\beta\gamma}}{\mu^2} \\
& + \left[2 \left[3\psi'(\tfrac{1}{3})\alpha + 3\psi'(\tfrac{1}{3}) - 2\pi^2\alpha - 2\pi^2 \right] \frac{a}{81} \right. \\
& \quad + \left[[912\pi^2 - 1368\psi'(\tfrac{1}{3})] N_f \right. \\
& \quad - [333\alpha + 162] \psi'''(\tfrac{1}{3}) \\
& \quad + [2322\alpha^2 - 1188\alpha + 31050] \psi'(\tfrac{1}{3}) \\
& \quad + [19440 - 104976\alpha] s_2(\tfrac{\pi}{6}) \\
& \quad \left. + [209952\alpha - 38880] s_2(\tfrac{\pi}{2}) \right]
\end{aligned}$$

$$\begin{aligned}
& + [174960\alpha - 32400] s_3(\frac{\pi}{6}) \\
& + [25920 - 139968\alpha] s_3(\frac{\pi}{2}) \\
& + [888\alpha + 432] \pi^4 + [4212\alpha + 7128] \Sigma \\
& - [20700 - 792\alpha + 1548\alpha^2] \pi^2 \\
& + [15552\alpha - 19764] \zeta(3) \\
& + [8748\alpha - 1620] \frac{\ln(3)\pi}{\sqrt{3}} \\
& + [135 - 729\alpha] \frac{\ln^2(3)\pi}{\sqrt{3}} \\
& + [783\alpha - 145] \frac{\pi^3}{\sqrt{3}} \left] \frac{a^2}{8748} \right] \frac{\sigma_{\alpha\delta}^{p\mu} \otimes \sigma_{p\mu\beta\gamma}}{\mu^2} \\
& + \left[4 \left[3\psi'(\frac{1}{3})\alpha + 3\psi'(\frac{1}{3}) - 2\pi^2\alpha - 2\pi^2 \right] \frac{a}{81} \right. \\
& + \left[[1968\pi^2 - 2952\psi'(\frac{1}{3})] N_f \right. \\
& \quad - [666\alpha + 765] \psi'''(\frac{1}{3}) \\
& \quad + [4644\alpha^2 - 2376\alpha + 30780] \psi'(\frac{1}{3}) \\
& \quad - [295488 + 209952\alpha] s_2(\frac{\pi}{6}) \\
& \quad + [419904\alpha + 590976] s_2(\frac{\pi}{2}) \\
& \quad + [349920\alpha + 492480] s_3(\frac{\pi}{6}) \\
& \quad - [393984 + 279936\alpha] s_3(\frac{\pi}{2}) \\
& \quad + [1776\alpha + 2040] \pi^4 + [8424\alpha + 11340] \Sigma \\
& \quad - [20520 - 1584\alpha + 3096\alpha^2] \pi^2 \\
& \quad + [31104\alpha - 21708] \zeta(3) \\
& \quad + [17496\alpha + 24624] \frac{\ln(3)\pi}{\sqrt{3}} \\
& \quad - [2052 + 1458\alpha] \frac{\ln^2(3)\pi}{\sqrt{3}} \\
& \quad \left. + [2204 + 1566\alpha] \frac{\pi^3}{\sqrt{3}} \right] \frac{a^2}{8748} \right] \frac{\sigma_{\alpha\delta}^{p\mu} \otimes \sigma_{q\mu\beta\gamma}}{\mu^2} \\
& + \left[\left[[212\psi'(\frac{1}{3}) - 144\pi^2] N_f + 441\psi'''(\frac{1}{3}) \right. \right. \\
& \quad + 31320\psi'(\frac{1}{3}) + 334368s_2(\frac{\pi}{6}) - 668736s_2(\frac{\pi}{2}) \\
& \quad - 557280s_3(\frac{\pi}{6}) + 445824s_3(\frac{\pi}{2}) - 1176\pi^4 \\
& \quad - 20880\pi^2 + 2916\Sigma - 17820\zeta(3) \\
& \quad - 2494\frac{\pi^3}{\sqrt{3}} - 27864\frac{\ln(3)\pi}{\sqrt{3}} \\
& \quad \left. + 2322\frac{\ln^2(3)\pi}{\sqrt{3}} \right] \frac{a^2}{8748} \right] \frac{\sigma_{\alpha\delta}^{q\mu} \otimes \sigma_{p\mu\beta\gamma}}{\mu^2} \\
& + O(a^3)
\end{aligned} \tag{5.1}$$

where symm denotes (2.24) and $\psi(z)$ is the derivative of the logarithm of the Euler Γ -function. Other various quantities are defined by

$$\Sigma = \mathcal{H}_{31}^{(2)} + \mathcal{H}_{43}^{(2)} \quad , \quad s_n(z) = \frac{1}{\sqrt{3}} \Im \left[\text{Li}_n \left(\frac{e^{iz}}{\sqrt{3}} \right) \right] \tag{5.2}$$

where $\text{Li}_n(z)$ is the polylogarithm function. We retain the notation of [13] here in defining the quantity Σ which is a linear combination of two harmonic polylogarithms, $\mathcal{H}_{31}^{(2)}$ and $\mathcal{H}_{43}^{(2)}$. More background to their appearance in the basic master integrals can be found in Appendix A of [13]. The theory for such harmonic polylogarithms was developed in [50]. These polylogarithms together with the other quantities such as $\zeta(3)$ and $\ln(3)$ emerge from the one and two loop master integrals, [46, 47, 48, 49]. Recently, the mathematics of these masters has been studied in the context of cyclotomic polynomials and harmonic polylogarithms in [51]. There an insight has been given for which particular polylogarithms and other such numbers will arise in the higher loop order master integrals. We also note that our convention in (5.1) is that when a Lorentz index is contracted with one of the external momenta p or q then the Lorentz index is replaced by the momentum to compactify notation.

We only include the expression for the Green's function containing \mathcal{O}_1 since that for \mathcal{O}_2 can be readily deduced by multiplying (5.1) by $\gamma^5 \otimes \gamma^5$. Then the sum and difference of these two cases will give respectively the finite parts of the Green's function of the eigen-operators of $\gamma_{ij}(a)$. However, we have checked that both are in agreement in four dimensions. This is because while we have derived the finite renormalization required to restore anti-commutativity of γ^5 in four dimensions that was essentially based on the two structures $I \otimes I$ and $\gamma^5 \otimes \gamma^5$. Aside from the fact that the 3-quark operators are renormalizable, the reason for this is that these channels ordinarily contain the divergences in ϵ . Therefore in keeping with the Larin method, [31], they are used to define the finite renormalization matrix. In choosing this procedure it transpires that the coefficients in the other channels are in agreement when the conventions on the charge conjugation matrix, C , are respected. Next in extracting (5.1) from our d -dimensional expression in addition to (2.15) we have to be careful in taking the four dimensional limit of generalized Γ -matrices where there are contractions with either or both of the external momenta p and q . So in addition to (2.15) we have used

$$\begin{aligned} \Gamma_{(4)}^{p\mu\nu\sigma} \otimes \Gamma_{(4)q\mu\nu\sigma} \Big|_{d=4} &= 6pq\gamma^5 \otimes \gamma^5 \\ \Gamma_{(4)}^{pq\mu\nu} \otimes \Gamma_{(4)pq\mu\nu} \Big|_{d=4} &= 2p^2q^2\gamma^5 \otimes \gamma^5 - 2(pq)^2\gamma^5 \otimes \gamma^5 \end{aligned} \quad (5.3)$$

for (2.22). The explicit values, (2.24) and (2.25), can be substituted in these general expressions. We also should comment on the structure of (5.1) in the various channels. One approach to extract the finite part might have been to project the coefficients of each Lorentz structure. While we did not follow that line one can observe from the final result that there are seven such channels at two loops in *four* dimensions. In d -dimensions, prior to taking the limit to four dimensions after renormalization, there are more than seven channels. Moreover, it seems that not all possible structures are generated and it would appear that that strategy would require more effort than is necessary. Also internally it would require the manipulation of long strings of γ -matrices after applying the transpose of a set of these matrices due to (2.2). This would slow the symbolic manipulation programmes.

While (5.1) represents the evaluation of the Green's function to two loops in $\overline{\text{MS}}$ analytically, for practical purposes the numerical value is more useful. Therefore, we have evaluated (5.1) to seven decimal places and find

$$\begin{aligned} \langle \psi_\alpha(p) \psi_\beta(q) \psi_\gamma(-p-q) \mathcal{O}_1 \delta(0) \rangle \Big|_{\text{symm}} &= [1 + [0.9894261\alpha + 0.9894261] a \\ &\quad + [41.5310566 + 6.7082190\alpha + 2.8995053\alpha^2 \\ &\quad - 3.9141771N_f] a^2] I_{\alpha\delta} \otimes I_{\beta\gamma} \\ &\quad - 1.6908864a^2 \gamma_{\alpha\delta}^5 \otimes \gamma_{\beta\gamma}^5 \\ &\quad + [[0.5208683\alpha + 0.5208683] a \end{aligned}$$

$$\begin{aligned}
& + [16.3956216 + 6.1021902\alpha + 1.8664447\alpha^2 \\
& \quad - 1.2732336N_f] a^2] \frac{I_{\alpha\delta} \otimes \sigma_{\beta\gamma}^{pq}}{\mu^2} \\
& - 0.3372607a^2 \frac{\gamma_{\alpha\delta}^5 \otimes (\gamma^5 \sigma^{pq})_{\beta\gamma}}{\mu^2} \\
& + [[0.2604341\alpha + 0.2604341] a \\
& \quad + [9.4012326 + 3.0510951\alpha + 0.9332223\alpha^2 \\
& \quad - 0.5498054N_f] a^2] \frac{\sigma_{\alpha\delta}^{p\mu} \otimes \sigma_{p\mu\beta\gamma}}{\mu^2} \\
& + [[0.5208683\alpha + 0.5208683] a \\
& \quad + [17.4304130 + 6.1021902\alpha + 1.8664447\alpha^2 \\
& \quad - 1.1864222N_f] a^2] \frac{\sigma_{\alpha\delta}^{p\mu} \otimes \sigma_{q\mu\beta\gamma}}{\mu^2} \\
& + [1.3720521 + 0.0868114N_f] a^2 \frac{\sigma_{\alpha\delta}^{q\mu} \otimes \sigma_{p\mu\beta\gamma}}{\mu^2} \\
& + O(a^3)
\end{aligned} \tag{5.4}$$

where we have used

$$\begin{aligned}
\zeta(3) &= 1.20205690 \quad , \quad \Sigma = 6.34517334 \quad , \quad \psi' \left(\frac{1}{3} \right) = 10.09559713 \quad , \\
\psi''' \left(\frac{1}{3} \right) &= 488.1838167 \quad , \quad s_2 \left(\frac{\pi}{2} \right) = 0.32225882 \quad , \quad s_2 \left(\frac{\pi}{6} \right) = 0.22459602 \quad , \\
s_3 \left(\frac{\pi}{2} \right) &= 0.32948320 \quad , \quad s_3 \left(\frac{\pi}{6} \right) = 0.19259341
\end{aligned} \tag{5.5}$$

as the input values in this exercise. Finally, we have not included the finite part of the Green's function (2.26) to two or three loops since lattice measurements would require a zero momentum quark. This is a very difficult task numerically on the lattice. Moreover, only the two loop results of (2.22) are in an arbitrary linear covariant gauge since we restricted the three loop MINCER calculation to the Feynman gauge.

6 General operator

We now consider a generalization of the basic spin $\frac{1}{2}$ operators we have focused on so far. Recently, Kränkl and Manashov, [32], have introduced the operator

$$\mathcal{O}_{\alpha\beta\gamma}^{ijk} = \epsilon^{IJK} \psi_{\alpha}^{iI} \psi_{\beta}^{jJ} \psi_{\gamma}^{kK} \tag{6.1}$$

which has no contractions over the spinor indices and i, j and k are flavour indices. By considering the renormalization of this basic operator and its mixing into operators of the same dimension they managed to derive two loop expressions for several other operator aside from the two considered in the previous sections. However, the anomalous dimensions recorded in [32] are not in the $\overline{\text{MS}}$ scheme. Despite this we have extended the results of [32] to three loops. This is straightforward as the symbolic manipulation programmes used to derive the $\overline{\text{MS}}$ renormalization of the $(\frac{1}{2}, 0)$ operators was sufficiently general and hence adaptable to (6.1). First, we recall the notation and formalism for the renormalization of (6.1), [32]. The bare operator mixes into

an infinite set of related operators which involve the generalized γ -matrices, $\Gamma_{(n)}^{\mu_1 \dots \mu_n}$. Though at each order in perturbation theory the number of generated operators is finite. Specifically,

$$\mathcal{O}_{\alpha\beta\gamma}^{ijk} = Z_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'} \mathcal{O}_{\alpha'\beta'\gamma'}^{ijk} \quad (6.2)$$

where the renormalization constant matrix $Z_{\alpha\beta\gamma}^{\alpha'\beta'\gamma'}$ is given by

$$Z_{\alpha\alpha'\beta\beta'\gamma\gamma'} = \delta_{\alpha\alpha'}\delta_{\beta\beta'}\delta_{\gamma\gamma'} + \sum_k a_{mnp}(\epsilon) \mathbb{G}_{mnp(\alpha\alpha'|\beta\beta'|\gamma\gamma')} . \quad (6.3)$$

Here the poles in ϵ are contained within the function $a_{mnp}(\epsilon)$ where $k \equiv mnp$ is a label which indicates the basic $\Gamma_{(n)}$ -matrix structure and there is no sum over individual m , n and p but over the corporate label k , and

$$a_{mnp}(\epsilon) = \sum_{n=1}^{\infty} \frac{a_{mnp}^{(n)}}{\epsilon^n} . \quad (6.4)$$

As there are three open spinor indices in (6.1) the $\Gamma_{(n)}$ -matrix structure is of the form, [32],

$$\mathbb{G}_{mnp(\alpha\alpha'|\beta\beta'|\gamma\gamma')} = \Gamma_{(m)\alpha\alpha'} \otimes \Gamma_{(n)\beta\beta'} \otimes \Gamma_{(p)\gamma\gamma'} . \quad (6.5)$$

We will omit spinor indices from this point and use tensor product notation as it is clearer. We have not included the Lorentz indices here but the contractions are across different Γ -matrices due to the antisymmetric property and there are no free Lorentz indices. It turns out that from explicit calculations the $\Gamma_{(n)}$ -structures appear in a symmetric form. To three loops, using the same notation as [32], these are

$$\begin{aligned} \mathbb{C}_0 &= \Gamma_{000} , \quad \mathbb{C}_2 = \Gamma_{220} + \Gamma_{202} + \Gamma_{022} , \quad \mathbb{C}_4 = \Gamma_{440} + \Gamma_{404} + \Gamma_{044} \\ \mathbb{C}_6 &= \Gamma_{660} + \Gamma_{606} + \Gamma_{066} , \quad \mathbb{C}_{222} = \Gamma_{222} , \quad \mathbb{C}_{42} = \Gamma_{422} + \Gamma_{242} + \Gamma_{224} \\ \mathbb{C}_{442} &= \Gamma_{442} + \Gamma_{424} + \Gamma_{244} , \quad \mathbb{C}_{444} = \Gamma_{444} \\ \mathbb{C}_{642} &= \Gamma_{642} + \Gamma_{624} + \Gamma_{462} + \Gamma_{426} + \Gamma_{246} + \Gamma_{264} \end{aligned} \quad (6.6)$$

where \mathbb{C}_i retain the same spinor index structure as \mathbb{G} . We have included \mathbb{C}_{222} and \mathbb{C}_{442} in this list as they appear at intermediate parts of the renormalization but are absent in the final expression in keeping with the expectation that the total number of γ -matrices should be divisible by four. To avoid any confusion the explicit contraction of the Lorentz indices in each definition of (6.6) is

$$\begin{aligned} \Gamma_{000} &= \Gamma_{(0)} \otimes \Gamma_{(0)} \otimes \Gamma_{(0)} \\ \Gamma_{220} &= \Gamma_{(2)}^{\mu_1\mu_2} \otimes \Gamma_{(2)\mu_1\mu_2} \otimes \Gamma_{(0)} \\ \Gamma_{440} &= \Gamma_{(4)}^{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(4)\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(0)} \\ \Gamma_{660} &= \Gamma_{(6)}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \otimes \Gamma_{(6)\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \otimes \Gamma_{(0)} \\ \Gamma_{222} &= \Gamma_{(2)}^{\mu_1\mu_2} \otimes \Gamma_{(2)\mu_1\mu_3} \otimes \Gamma_{(2)\mu_2}^{\mu_3} \\ \Gamma_{422} &= \Gamma_{(4)}^{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(2)\mu_1\mu_2} \otimes \Gamma_{(2)\mu_3\mu_4} \\ \Gamma_{442} &= \Gamma_{(4)}^{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(4)\mu_1\mu_2\mu_3}^{\mu_5} \otimes \Gamma_{(2)\mu_4\mu_5} \\ \Gamma_{444} &= \Gamma_{(4)}^{\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(4)\mu_1\mu_2}^{\mu_5\mu_6} \otimes \Gamma_{(4)\mu_3\mu_4\mu_5\mu_6} \\ \Gamma_{642} &= \Gamma_{(6)}^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} \otimes \Gamma_{(4)\mu_1\mu_2\mu_3\mu_4} \otimes \Gamma_{(2)\mu_5\mu_6} \end{aligned} \quad (6.7)$$

with the obvious permutation of m , n and p to define the forms in (6.6) not listed in (6.7). In (6.6) \mathbb{C}_0 , \mathbb{C}_2 , \mathbb{C}_4 and \mathbb{C}_{42} arise at two loops, [32], and \mathbb{C}_6 , \mathbb{C}_{222} , \mathbb{C}_{442} , \mathbb{C}_{444} and \mathbb{C}_{642} only appear

at three loops. That there are no other structures to this order is elementary to deduce from the fact that beginning with (6.1) there are 4, 8 and 12 possible γ -matrices in each of the respective one, two and three loop Feynman diagrams. With the absence of free Lorentz indices and the antisymmetry (6.6) are all that survive.

In [32] in order to ease the derivation of the anomalous dimension of (6.1) at two loops in four dimensions a relation was derived for \mathbb{C}_{42} in d -dimensions which was

$$\mathbb{C}_{42} = -3d(d-1)\mathbb{C}_0 - 2(d-3)\mathbb{C}_2 - \frac{1}{2}\mathbb{C}_4 + \frac{1}{2}\mathbb{C}_2^2 \quad (6.8)$$

where the product of the \mathbb{C}_l is regarded as the multiplication of the constituent γ -matrices. While this is a relation in d -dimensions ultimately we will require the anomalous dimension in four dimensions and as noted in [32] then

$$\mathbb{C}_4 \Big|_{d=4} = 24 [\gamma^5 \otimes \gamma^5 \otimes I + \gamma^5 \otimes I \otimes \gamma^5 + I \otimes \gamma^5 \otimes \gamma^5] . \quad (6.9)$$

In addition we have the similar but more trivial relations

$$\begin{aligned} \mathbb{C}_0 \Big|_{d=4} &= I \otimes I \otimes I \\ \mathbb{C}_2 \Big|_{d=4} &= [\sigma^{\mu\nu} \otimes \sigma_{\mu\nu} \otimes I + \sigma^{\mu\nu} \otimes I \otimes \sigma_{\mu\nu} + I \otimes \sigma^{\mu\nu} \otimes \sigma_{\mu\nu}] . \end{aligned} \quad (6.10)$$

At three loops two new structures emerge, \mathbb{C}_6 and \mathbb{C}_{642} , which involve the evanescent $\Gamma_{(6)}^{\mu_1 \dots \mu_6}$ matrix. However, similar to (6.8) one can deduce that in d -dimensions

$$\begin{aligned} \mathbb{C}_6 &= -12d(d-1)(2d-1)\mathbb{C}_0 - 3(d-1)(7d-24)\mathbb{C}_2 - 6(2d-5)\mathbb{C}_4 \\ &\quad + 2(3d-4)\mathbb{C}_2^2 - \frac{1}{2}\mathbb{C}_2^3 + \frac{3}{2}\mathbb{C}_2\mathbb{C}_4 + 3\mathbb{C}_{444} \\ \mathbb{C}_{642} &= 12d(d-1)(2d-7)\mathbb{C}_0 + 9(d^2-9d+16)\mathbb{C}_2 + 2(2d-5)\mathbb{C}_4 \\ &\quad - 2(3d-10)\mathbb{C}_2^2 + \frac{1}{2}\mathbb{C}_2^3 - \frac{1}{2}\mathbb{C}_2\mathbb{C}_4 - 3\mathbb{C}_{444} . \end{aligned} \quad (6.11)$$

So these evanescent combinations can be expressed in terms of \mathbb{C}_i which do not involve any $\Gamma_{(n)}^{\mu_1 \dots \mu_n}$ with $n \geq 5$. For the restriction to four dimensions we have the additional relation

$$\mathbb{C}_{444} \Big|_{d=4} = 0 . \quad (6.12)$$

This follows trivially from the antisymmetry property. Given the presence of $\Gamma_{(4)}^{\mu_1 \mu_2 \mu_3 \mu_4}$ one possibility for this could have been $\gamma^5 \otimes \gamma^5 \otimes \gamma^5$. It is easy to see that this is excluded when one examines the pattern of Lorentz indices in four dimensions. The use of the relations (6.11) can be viewed within the approach of [32] as a variation of the formalism of [30] which was introduced to include the effect of evanescent operators in the renormalization group functions.

Equipped with these identities we have extracted the three loop anomalous dimensions (6.1) in the $\overline{\text{MS}}$ scheme from the same three loop MINCER computation as in previous sections. Though in this case we do not contract the free spinor indices to produce a spin $\frac{1}{2}$ operator. The full result is

$$\begin{aligned} \gamma_{\mathcal{O}}(a) &= -\frac{1}{6}\mathbb{C}_2 a + \left[[36 - 2N_f]\mathbb{C}_0 + \left[\frac{1}{54}N_f - \frac{47}{36} \right] \mathbb{C}_2 - \frac{1}{72}\mathbb{C}_2^2 + \frac{5}{72}\mathbb{C}_4 \right] a^2 \\ &\quad + \left[\left[\frac{10}{9}N_f^2 - \frac{853}{9}N_f + \frac{8047}{9} + 17\zeta(3) \right] \mathbb{C}_0 \right. \end{aligned}$$

$$\begin{aligned}
& + \left[\frac{433}{36} \zeta(3) - \frac{5873}{216} + \left[\frac{71}{54} + \frac{20}{9} \zeta(3) \right] N_f + \frac{13}{162} N_f^2 \right] \mathbb{C}_2 \\
& + \left[\frac{209}{648} - \frac{71}{54} \zeta(3) - \frac{1}{648} N_f \right] \mathbb{C}_2^2 + \left[\frac{1}{54} \zeta(3) - \frac{5}{1296} \right] \mathbb{C}_2^3 \\
& + \left[\frac{29}{24} \zeta(3) - \frac{91}{144} - \frac{7}{648} N_f \right] \mathbb{C}_4 + \left[\frac{37}{864} - \frac{25}{288} \zeta(3) \right] \mathbb{C}_2 \mathbb{C}_4 \\
& + \left[\frac{1}{16} - \frac{1}{9} \zeta(3) \right] \mathbb{C}_{444} \Big] a^3 + O(a^4) .
\end{aligned} \tag{6.13}$$

The two loop part is in exact agreement with [32]. Moreover, our two loop computation was carried out in an arbitrary linear covariant gauge and we observed the cancellation of the gauge parameter which provides an additional check. The three loop diagrams were computed in the Feynman gauge and the double and triple poles in ϵ in the three loop renormalization constant satisfy the underlying renormalization group formalism as otherwise a finite expression would not have emerged. Therefore we are confident that (6.13) correctly extends the result of [32]. If one restricted to four dimensions then the final term involving \mathbb{C}_{444} would be absent and the corresponding four dimensional expressions for the remaining \mathbb{C}_i used.

Spin	Chirality	\mathbb{C}_0	\mathbb{C}_2	\mathbb{C}_4	\mathbb{C}_{444}
$(\frac{1}{2}, 0)$	+	1	12	72	0
$(\frac{1}{2}, 0)$	-	1	12	-24	0
$(\frac{3}{2}, 0)$	+	1	-12	72	0
$(1, \frac{1}{2})$	-	1	-4	-24	0

Table 1. Values for the evaluation of the general anomalous dimension for various nucleons.

Equipped with the general anomalous dimension we can extend the two loop results in the renormalization scheme of [32] for eigen-operators with specific spins and chirality. As indicated in [32] these relate to various operators in the literature. If we denote the label which the irreducible representations of the Lorentz group with two spins j and \bar{j} by (j, \bar{j}) have then the eigen-operators are $\mathcal{O}^{(j, \bar{j})}$. They can be written in the forms, [23, 32, 38, 39],

$$\begin{aligned}
\mathcal{O}_+^{(\frac{1}{2}, 0)} &= \epsilon^{IJK} \psi_L^I \left((\psi_L^J)^T C \psi_L^K \right) \quad , \quad \mathcal{O}_-^{(\frac{1}{2}, 0)} = \epsilon^{IJK} \psi_R^I \left((\psi_L^J)^T C \psi_L^K \right) \\
\mathcal{O}_+^{(\frac{3}{2}, 0)} &= \epsilon^{IJK} \Delta \psi_L^I \Delta \psi_L^J \Delta \psi_L^K \quad , \quad \mathcal{O}_+^{(1, \frac{1}{2})} = \epsilon^{IJK} \Delta \psi_L^I \Delta \psi_L^J \Delta \psi_R^K .
\end{aligned} \tag{6.14}$$

where $\Delta^2 = 0$. Here we have denoted right and left handed quarks by $\psi_R = \frac{1}{2}(1 + \gamma^5)\psi$ and $\psi_L = \frac{1}{2}(1 - \gamma^5)\psi$ respectively. To determine the anomalous dimensions of each operator from the general anomalous dimension we replace these four dimensional tensor product matrices \mathbb{C}_i by their eigenvalue under the Lorentz symmetry, [32]. These have been given in [32] but are summarized in Table 1. Hence we have

$$\begin{aligned}
\gamma_+^{(\frac{1}{2}, 0)}(a) &= -2a + \left[\frac{70}{3} - \frac{16}{9} N_f \right] a^2 + \left[\frac{56}{27} N_f^2 + \left[\frac{80}{3} \zeta(3) - 80 \right] N_f + \frac{5392}{9} + 16 \zeta(3) \right] a^3 \\
&+ O(a^4) \\
\gamma_-^{(\frac{1}{2}, 0)}(a) &= -2a + \left[\frac{50}{3} - \frac{16}{9} N_f \right] a^2 + \left[\frac{56}{27} N_f^2 + \left[\frac{80}{3} \zeta(3) - \frac{2132}{27} \right] N_f + \frac{5494}{9} \right] a^3 + O(a^4) \\
\gamma_+^{(\frac{3}{2}, 0)}(a) &= 2a + \left[\frac{164}{3} - \frac{20}{9} N_f \right] a^2 + \left[\frac{4}{27} N_f^2 - \left[\frac{1004}{9} + \frac{80}{3} \zeta(3) \right] N_f + 1191 - \frac{560}{3} \zeta(3) \right] a^3 \\
&+ O(a^4)
\end{aligned}$$

$$\begin{aligned}\gamma_-^{(1, \frac{1}{2})}(a) &= \frac{2}{3}a + \left[\frac{118}{3} - \frac{56}{27}N_f \right] a^2 + \left[\frac{64}{81}N_f^2 - \left[\frac{8084}{81} + \frac{80}{9}\zeta(3) \right] N_f + \frac{9248}{9} - \frac{272}{3}\zeta(3) \right] a^3 \\ &\quad + O(a^4) .\end{aligned}\tag{6.15}$$

To assist with a comparison the numerical values are

$$\begin{aligned}\gamma_+^{(\frac{1}{2}, 0)}(a) &= -2.0000000a + [23.3333333 - 1.7777778N_f] a^2 \\ &\quad + [2.0740741N_f^2 - 47.9451492N_f + 618.3440216] a^3 + O(a^4) \\ \gamma_-^{(\frac{1}{2}, 0)}(a) &= -2.0000000a + [16.6666667 - 1.7777778N_f] a^2 \\ &\quad + [2.0740741N_f^2 - 46.9081122N_f + 610.4444444] a^3 + O(a^4) \\ \gamma_+^{(\frac{3}{2}, 0)}(a) &= 2.0000000a + [54.6666667 - 2.2222222N_f] a^2 \\ &\quad + [0.1481481N_f^2 - 143.6104063N_f + 966.6160447] a^3 + O(a^4) \\ \gamma_-^{(1, \frac{1}{2})}(a) &= 0.6666667a + [39.3333333 - 2.0740741N_f] a^2 \\ &\quad + [0.7901235N_f^2 - 110.4874194N_f + 918.5690630] a^3 + O(a^4) .\end{aligned}\tag{6.16}$$

Clearly the coefficients of the anomalous dimension derived from the generalized operator approach of [32] are larger in value than our $\overline{\text{MS}}$ direct calculation. Therefore it would appear that the latter anomalous dimensions have a slower rate of convergence.

Comparing the expressions for the $(\frac{1}{2}, 0)$ pair of operators to our $\overline{\text{MS}}$ expressions we see that the one loop terms are the same. This is expected since that part of an anomalous dimension is scheme independent. The two and three loop terms are not the same. This difference is due to renormalization scheme dependence. While the subtraction method used is in principle the same in both cases since only the poles in ϵ are removed into renormalization constants, it is in the derivation of these poles in the d -dimensional calculations where the differences arise. Moreover, in [32] the issue of having to handle γ^5 in dimensional regularization is circumvented unlike our extension of the two loop $\overline{\text{MS}}$ computation of [20] where we completely reproduced that result. One issue relating to this concerns whether it is possible to derive (4.8) from the general operator anomalous dimension, (6.13). A clue resides in the comparison of the expression for both chiralities of the spin $\frac{1}{2}$ operators. If we compute the ratio of the anomalous dimensions of the chiralities for both spin $\frac{1}{2}$ cases we find

$$\begin{aligned}\frac{\gamma_+^{(\frac{1}{2}, 0)}(a)}{\gamma_+(a)} &= 1 + \frac{7}{18}[2N_f - 33]a - \frac{1}{162}[52N_f^2 - 4194N_f + 34821] a^2 + O(a^3) \\ \frac{\gamma_-^{(\frac{1}{2}, 0)}(a)}{\gamma_+(a)} &= 1 + \frac{7}{18}[2N_f - 33]a - \frac{1}{162}[52N_f^2 - 3774N_f + 27891] a^2 + O(a^3) .\end{aligned}\tag{6.17}$$

As in [32] the first two terms are in agreement but differ now in the $O(a^2)$ terms. Moreover, the coefficient of the one loop β -function appears in the discrepancy between the $\overline{\text{MS}}$ result of (4.8) and (6.15). This can be explained by recalling that in the mapping of the general result \mathbb{C}_2 is replaced by 12. However, this is the four dimensional evaluation of $d(d-1)$ which is derived from the product $\Gamma_{(2)}^{\mu\nu}\Gamma_{(2)\mu\nu}$. Expressing this in terms of ϵ gives

$$d(d-1) = 12 \left[1 - \frac{7}{6}\epsilon + \frac{1}{3}\epsilon^2 \right] .\tag{6.18}$$

Such a factor emerges from \mathbb{C}_2 when one projects (6.13) onto the proton operator by using the formal contraction $I \otimes I$. Therefore, including the contribution from the $O(\epsilon)$ term within

the mapping used in [32] reproduces the corresponding two loop terms of (4.8). To extend this to the next order is certainly highly non-trivial. This is partly because there are more terms corresponding to additional operators in (6.13) but also due to the now hidden general evanescent operators as well as the γ^5 issue. The operators are hidden in the sense that their effect in d -dimensions cannot be restored from the explicit expression in (6.13). More crucially, though, in order to proceed along these lines one is in effect repeating the actual calculation anyway which was used to derive (4.8) in the first place. Indeed an analogous analysis for 4-fermi operators, [52], only serves to illustrate the large complexity of such a problem which is beyond the scope of the present article.

7 Discussion.

By way of concluding remarks we note that first we have extended the two loop $\overline{\text{MS}}$ renormalization of the 3-quark proton operator, [20], to three loops. This is a technically more involved computation than [20] since the γ^5 problem in dimensional regularization can no longer be treated passively at three loops. To accommodate this we have extended Larin's method for automatic symbolic manipulation programmes to operators which mix under renormalization. Similar features to [31] emerge in that the finite renormalization constant which is required to restore anti-commutativity in four dimensions is independent of the gauge parameter. We have indicated that this is true to all orders if the naive anomalous dimensions of the operators are independent of the gauge parameter which is the case for $\overline{\text{MS}}$. By contrast in other mass dependent renormalization schemes this finite renormalization would be gauge dependent. The reasoning for this is that when a gauge invariant operator is renormalized in a mass dependent scheme its anomalous dimension depends on the gauge parameter. In addition what is apparent from comparing the various finite renormalization constants used to restore chirality in four dimensions both here and in [31] is that there is no *universal* finite renormalization. In other words one cannot merely extract a Z^5 from [31] and use it within another computation where the seed operator is not even present. While it may appear to be satisfactory at a particular low loop order for a Green's function it will lead to inconsistencies at higher loop order. In other words for each appearance of γ^5 within an automatic symbolic manipulation computation one has to systematically treat γ^5 in an adaptation of the Larin method.

Moving away from the above general remarks concerning γ^5 we note that the 3-quark operator we concentrated on was that relating to the proton or $(\frac{1}{2}, 0)$ in Lorentz spin notation. Ideally other spin operators are of interest and we have provided a first step in that direction by extending the recent analysis of [32] to three loops. This used a generalized operator approach which resolved the evanescent and γ^5 issues from another angle. Though as we have remarked it is clear that the results are not in the $\overline{\text{MS}}$ scheme if one regards the earlier work of [31] as the true $\overline{\text{MS}}$ situation which we are in agreement with. To extend the 3-quark operator $\overline{\text{MS}}$ renormalization to other spin operators is not straightforward. This is because there is mixing between operators which is dependent on flavour symmetry. Our computational setup was designed purely for the proton case and will need to be extended to accommodate these other operators which is a topic we hope to return to later. The explicit definition of these operators and the relation to the flavour structure is given in [24, 39]. Whilst the motivation for this work is in relation to providing the perturbative structure of Green's functions to assist lattice matching in the high energy limit, in order to refine the understanding of proton structure will require an extension of our analysis in another direction. In essence this involves the treatment of operators with higher moments which manifest themselves in the decoration of (2.1) with covariant derivatives. For instance, a three loop renormalization of the first moment will require

the order of an additional 700 Feynman diagrams to be calculated. Again we hope to return to this in a later analysis.

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References.

- [1] G.P. Lepage & S.J. Brodsky, Phys. Rev. Lett. **43** (1979), 545.
- [2] G.P. Lepage & S.J. Brodsky, Phys. Rev. **D22** (1980), 2157.
- [3] M.E. Peskin, Phys. Lett. **B88** (1979), 128.
- [4] D.J. Gross & F.J. Wilczek, Phys. Rev. **D9** (1974), 980.
- [5] E.G. Floratos, D.A. Ross & C.T. Sachrajda, Nucl. Phys. **B129** (1977), 66; **B139** (1978), 545(E).
- [6] E.G. Floratos, D.A. Ross & C.T. Sachrajda, Nucl. Phys. **B152** (1979), 493.
- [7] S.G. Gorishny, S.A. Larin, L.R. Surguladze & F.K. Tkachov, Comput. Phys. Commun. **55** (1989), 381.
- [8] S.A. Larin, F.V. Tkachov & J.A.M. Vermaseren, “The Form version of Mincer”, NIKHEF-H-91-18.
- [9] K.G. Chetyrkin & A. Rétey, Nucl. Phys. **B583** (2000), 3.
- [10] J.A. Gracey, Nucl. Phys. **B662** (2003), 247.
- [11] C. Sturm, Y. Aoki, N.H. Christ, T. Izubuchi, C.T.C. Sachrajda & A. Soni, Phys. Rev. **D80** (2009), 014501.
- [12] M. Gorbahn & S. Jäger, Phys. Rev. **D82** (2010), 114001.
- [13] L.G. Almeida & C. Sturm, Phys. Rev. **D82** (2010), 054017.
- [14] J.A. Gracey, Eur. Phys. J. **C71** (2011), 1567.
- [15] J.A. Gracey, JHEP **1103** (2011), 109.
- [16] G. Martinelli, C. Pittori, C.T. Sachrajda, M. Testa & A. Vladikas, Nucl. Phys. **B445** (1995), 81.
- [17] E. Franco & V. Lubicz, Nucl. Phys. **B531** (1998), 641
- [18] D.G. Richards, C.T. Sachrajda & C.J. Scott, Nucl. Phys. **B286** (1987), 683.
- [19] M. Göckeler, R. Horsley, T. Kaltenbrunner, Y. Nakamura, D. Pleiter, P.E.L. Rakow, A. Schäfer, G. Schierholz, H. Stüben, N. Warkentin & J.M. Zanotti, Nucl. Phys. **B812** (2009), 205.
- [20] A.A. Pivovarov & L.R. Surguladze, Nucl. Phys. **B360** (1991), 97.

- [21] A.G. Grozin & O.I. Yakovlev, Phys. Lett. **B285** (1992), 254.
- [22] S. Groote, J.G. Körner & O.I. Yakovlev, Phys. Rev. **D54** (1996), 3447.
- [23] B.L. Ioffe, Nucl. Phys. **B188** (1981), 317; Nucl. Phys. **B191** (1981), 591.
- [24] D. Espriu, P. Pascual & R. Tarrach, Nucl. Phys. **B214** (1983), 285.
- [25] Y. Chung, H.G. Dosch, M. Kremer & D. Schall, Nucl. Phys. **B197** (1982), 55.
- [26] V.M. Belyaev & B.L. Ioffe, Sov. Phys. JETP **56** (1982), 493.
- [27] Y. Chung, H.G. Dosch, M. Kremer & D. Schall, Z. Phys. **C15** (1982), 367.
- [28] B.L. Ioffe, Z. Phys. **C18** (1983), 67.
- [29] L.J. Reinders, H. Rubinstein & S. Yazaki, Phys. Rept. **127** (1985), 1.
- [30] A. Bondi, G. Curci, G. Paffuti & P. Rossi, Ann. Phys. **199** (1990), 268.
- [31] S.A. Larin, Phys. Lett. **B303** (1993), 113.
- [32] S. Kränkl & A. Manashov, Phys. Lett. **B703** (2011), 519.
- [33] M.J. Dugan & B. Grinstein, Phys. Lett. **B256** (1991), 239.
- [34] A.D. Kennedy, J. Math. Phys. **22** (1981), 1330.
- [35] A.N. Vasil'ev, S.É. Derkachov & N.A. Kivel, Theor. Math. Phys. **103** (1995), 487.
- [36] A.N. Vasil'ev, M.I. Vyazovskii, S.É. Derkachov & N.A. Kivel, Theor. Math. Phys. **107** (1996), 441.
- [37] A.N. Vasil'ev, M.I. Vyazovskii, S.É. Derkachov & N.A. Kivel, Theor. Math. Phys. **107** (1996), 710.
- [38] J.A.M. Vermaseren, math-ph/0010025.
- [39] Y. Chung, H.G. Dosch, M. Kremer & D. Schall, Z. Phys. **C25** (1984), 151.
- [40] J.A. Gracey, paper in preparation.
- [41] P. Nogueira, J. Comput. Phys. **105** (1993), 279.
- [42] M. Tentyukov & J.A.M. Vermaseren, Comput. Phys. Commun. **181** (2010), 1419.
- [43] S. Laporta, Int. J. Mod. Phys. **A15** (2000), 5087.
- [44] C. Studerus, Comput. Phys. Commun. **181** (2010), 1293.
- [45] C.W. Bauer, A. Frink & R. Kreckel, cs/0004015.
- [46] A.I. Davydychev, J. Phys. **A25** (1992), 5587.
- [47] N.I. Usyukina & A.I. Davydychev, Phys. Atom. Nucl. **56** (1993), 1553.
- [48] N.I. Usyukina & A.I. Davydychev, Phys. Lett. **B332** (1994), 159.
- [49] T.G. Birthwright, E.W.N. Glover & P. Marquard, JHEP **0409** (2004), 042.

- [50] E. Remiddi & J.A.M. Vermaseren, Int. J. Mod. Phys. **A15** (2000), 725.
- [51] J. Ablinger, J. Blümlein & C. Schneider, J. Math. Phys. **52** (2011), 102301.
- [52] K. Chetyrkin, M. Misiak & M. Münz, Nucl. Phys. **B520** (1998), 279.